Scalar cosmological perturbation in an inflationary brane world driven by the bulk inflaton

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Abstract

We investigate scalar perturbations from inflation in a bulk inflaton braneworld model. Using the generalized longitudinal gauge, we derive and solve the full set of scalar perturbation equations. Our exact results support the recent argument that for the de Sitter brane the square of the radion mass is not positive, showing that unlike the flat brane case, the de Sitter brane is not stable.

PACS numbers: 98.80.Cq, 04.50.+h

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I. INTRODUCTION

Based mostly upon string theory, the brane world scenario emerged in the past few years and offered dramatic changes in our thinking about the universe (for a review see [1] and references therein). It is widely believed that we may live on a hypersurface in higher dimensions with ordinary matter fields being confined on the brane and graviton propagating through extra dimensions. For the brane world scenario to become mature and successful, it has to confront experimental tests. Cosmological perturbations may be the main avenue to probe the brane world models. The recent accurate CMB experiment and expected promising future precision experiments will provide a great deal of information on the cosmological perturbations and test the brane world model observationally. With this motivation, theoretically it is of great interest to clarify the cosmological perturbations in brane worlds.

There have been a lot of papers studying static geometries with branes, including flat stabilized branes as well as curved de Sitter branes. The theory of scalar fluctuations around flat stabilized branes, involving bulk scalar field fluctuations, scalar bulk metric fluctuations and brane displacement is well understood [2][3][4][5]. The system of dynamical equations can be diagonalized and the extra dimensional contribution can be separated out. The problem is reduced to solving a second-order differential equation of the extra dimensional contribution to the fluctuation satisfying the boundary condition at the brane. The massive spectrum of the scalar perturbation appears due to the extra dimensions. Tensor perturbations of the bulk geometry with stabilized branes have also been studied (see [6] for a general discussion).

The bulk geometry with curved de Sitter branes can be used to explain the inflation in the early universe. Inflation can occur by the dynamics of inflaton either on the brane or in the bulk. The theory of metric fluctuations around bulk geometry with inflating branes is more complicated than that for the flat branes. For the tensor fluctuations, only massless modes can be generated from inflation [7]. The study of the scalar perturbations in the braneworld inflation driven by the inflaton on the brane has been carried out by many authors [8] and it was found that inflation occurs basically in the same way as that of the ordinary four-dimensional universe. For the brane inflation driven by the bulk inflaton, the investigation is rather complicated. This is partly because the back-reaction to the bulk

geometry should be included in treating the dynamics of the bulk inflation [9], and partly because the bulk perturbation should be taken into account and the full five-dimensional Einstein equations have to be solved. In many works, perturbations of the inflation were considered by neglecting metric perturbations [10]. Recently, progress on this issue has been achieved. In [11], the cosmological perturbations in bulk inflaton model has been investigated by using a covariant curvature formalism. Further, choosing a specific bulk inflaton model with an exponential potential and dilatonic coupling to the brane tension, exact analytic solutions for the scalar perturbations have been obtained [12].

In this paper we are going to extend the study set up in [12] to another bulk inflaton model with a tachyonic bulk potential having a maximum at $\phi = 0$ and without coupling to brane tension [13]. With the generalized longitudinal gauge [2] for scalar perturbations, we derive the full set of scalar perturbation equations in the bulk inflaton model and obtain their analytic solutions. Using these analytic solutions, we find that in contrast to the flat brane case, for de Sitter branes the square of the eigenvalue of the scalar fluctuations is typically negative. This result was first got by a full nonlinear numerical treatment [14] and subsequently confirmed by a subsequent analysis [15]. Without taking any simplification on boundary conditions and bulk perturbation equations, our exact analytic result for the specific model can be used as a support to these arguments.

Our paper is organized as follow. In Section 2 we briefly review the geometrical background. In section 3 we give the full set of scalar perturbation equations and the boundary conditions. In Section 4 we solve the equations analytically. The last section is devoted to the summary and discussions. For convenience we also give some detailed derivations in the appendix.

Throughout the paper we adopt the conventions as follows: Capital Latin indices mark the bulk, $A, B, C \cdots = 0, 1, 2, 3, 5$(here 5 represent the extra dimension). The usual four dimensional space-time is signed by Greek indices running from 0 to 3. And we use lower case Latin indices ($i, j, k \cdots = 1, 2, 3$) for 3-brane. We denote the bulk coordinate by $X^A = (t, x^i, r)$ and use the metric signature $(+, -, -, -, -)$.

II. BACKGROUND

We consider a spacetime with a negative cosmological constant Λ_5 as well as a small scalar field ϕ inhabited in the bulk while the single brane is empty except for a positive tension σ_0 which is minimally coupled with the scalar field. We begin with an action given by [13]

$$\mathcal{S} = \int d^5x\sqrt{|g|}\Big(-\frac{1}{2k_5^2}(\mathcal{R}+2\Lambda_5) + \frac{1}{2}(\nabla^C\phi)^2 - V(\phi)\Big) - \int d^4x\sqrt{|\gamma|}\sigma_0 \tag{1}$$

here k_5^2 is the five-dimensional gravity constant. To recover the Randall-Sundrum model, we need to tune the tension $\sigma_0 = \sqrt{-6\Lambda_5/k_5^4}$ in the absence of the bulk scalar field. The four-dimensional hypersurface locates at $r = r_0$ and the induced metric reads

$$\gamma_{\mu\nu} = \partial_\mu X^A \partial_\nu X^B g_{AB} = \delta_\mu^A \delta_\nu^B g_{AB}$$

If we set $k_5^2 = 1$, the Einstein equation is

$$\begin{aligned} G^A{}_B - \Lambda_5 g^A{}_B = \partial^A\phi\partial_B\phi - g^A{}_B\Big(\frac{1}{2}(\nabla^C\phi)^2 - V(\phi)\Big) \\ + \frac{\sqrt{|\gamma|}}{\sqrt{|g|}}\gamma_{\mu\nu}\delta_B^\mu\delta_C^\nu g^{AC}\sigma_0\delta(r-r_0) \end{aligned} \tag{2}$$

and the scalar field equation is

$$\nabla_C\nabla^C\phi + \frac{\partial V(\phi)}{\partial\phi} = 0 \tag{3}$$

In this work, we will concentrate on the potential taking a tachyonic form $V(\phi) = V_0 + \frac{1}{2}M^2\phi^2$, with $V_0 > 0, M^2 < 0$ in the vicinity of $\phi = 0$. There exists a maximum of the potential somewhere at $\phi = \phi_{min} \approx 0$ at which $V(\phi_{min}) = V_0$, where the Randall-Sundrum(RS) type II flat brane is recovered if $V_0 = 0$. The non-vanishing V_0 would lead to the deviation from RS model. We assume that V is positive and may vary very slowly in space and time. The sufficiently slowly varying of the bulk scalar field will lead to the standard slow-roll inflation. The effective bulk cosmological constant is $\Lambda_{5,eff} = \Lambda_5 + V_0$ and we need $V_0 < |\Lambda_5|$ to ensure $\Lambda_{5,eff} < 0$. The curvature radius is $\ell = \sqrt{-\Lambda_{5,eff}/6}$. The bulk metric is

$$\begin{aligned} ds^2 &= g_{AB}dX^A dX^B \\ &= (H\ell)^2\sinh^2 y\Big(dt^2 - H^{-2}e^{2Ht}\delta_{ij}dx^i dx^j\Big) - \ell^2 dy^2 \quad (y \le y_0) \end{aligned} \tag{4}$$

where we have defined $y = r/\ell$ for simplicity. As a direct result, the brane position is determined by [13]

$$H(\ell) = \frac{1}{\ell \sinh y_0} \tag{5}$$

At the end of this section we introduce a new coordinate $\widetilde{X}^A = (u, x^i, v)$. The transformation $X^A \to \widetilde{X}^A$ can be realized by

$$\begin{aligned} u &= \frac{\cosh y}{\sinh y} e^{-Ht} \\ v &= \frac{1}{\sinh y} e^{-Ht} \\ x^i &= x^i \end{aligned} \quad , \tag{6}$$

We can rewrite the background metric (4) into the new form

$$ds^2 = a^2(v)\Big[du^2 - \delta_{ij}dx^i dx^j - dv^2\Big] \tag{7}$$

where we have defined $a^2(v) \equiv (\ell/v)^2$. In the pseudo-spacetime $\widetilde{X}^A$, it seems just like a RS world except that the brane is "moving". The simpler geometry in the new coordinate $\widetilde{X}^A$ will be shown convenient to solve the perturbations in the bulk since it is possible to diagonalize the perturbation equations. Such scalar models for a domain wall have been discussed in the literature in detail also by [16] and in another context by [17].

III. SCALAR PERTURBATION EQUATIONS AND BOUNDARY CONDITIONS

The scalar perturbations in brane world inflation with bulk scalar is very complicated. Indeed, one has to consider five dimensional scalar metric fluctuations and brane displacement induced by the bulk scalar field fluctuation. Taking the generalized longitudinal gauge for scalar perturbations, we have the perturbed metric in different coordinate systems X^A and $\widetilde{X}^A$. In the real spacetime $X^A = (t, x^i, y)$, the perturbed metric can be written as

$$\begin{aligned} ds^2 = (H\ell)^2 \sinh^2 y \Big[(1+2\Phi)dt^2 - H^{-2}e^{2Ht}(1-2\Psi)\delta_{ij}dx^i dx^j\Big] \\ -2B\ell dt dy - \ell^2(1-2N)dy^2 \quad , \end{aligned} \tag{8}$$

while in the pseudo-spacetime $\widetilde{X}^A = (u, x^i, v)$, we have the perturbed metric

$$ds^2 = a^2(v)\Big[(1+2\varphi)du^2 - 2W du dv - (1-2\psi)\delta_{ij}dx^i dx^j - (1-2\Gamma)dv^2\Big] \quad . \tag{9}$$

By using (6), it is easy to find that the relations between (Φ, Ψ, B, N) and $(\varphi, \psi, W, \Gamma)$ are

$$\begin{aligned}
\Phi &= \coth^2 y\varphi + \frac{1}{\sinh^2 y}\Gamma - \frac{\cosh y}{\sinh^2 y}W \quad , \\
N &= \frac{1}{\sinh^2 y}\varphi + \coth^2 y\Gamma - \frac{\cosh y}{\sinh^2 y}W \quad , \\
B &= \frac{1+\cosh^2 y}{\sinh y}H\ell W - 2\coth y H\ell(\varphi + \Gamma) \quad , \\
\Psi &= \psi \quad .
\end{aligned} \tag{10}$$

After rescaling the five-dimensional cosmological constant, we can take the perturbed source term $\delta T^A{}_B$ as well as the energy momentum tensor itself $T^A{}_B$ vanishing at the lowest order approximation. So the perturbed Einstein equations are simplified to be

$$\delta G^A{}_B = 0 \tag{11}$$

Although Einstein equations (11) are independent of coordinate, the boundary conditions(or the differentiation) may differ in coordinate X^A and $\widetilde{X}^A$. In the coordinate system $\widetilde{X}^A$, due to the maximally symmetric bulk spacetime (7), it is possible to find general solutions for perturbations in the bulk. As shown in [3][12], it can make perturbation equations diagonal and find general solutions easily.

A. Perturbation Equations

In this subsection we will work in the coordinate system $\widetilde{X}^A$ in order to derive perturbation equations. In the absence of bulk sources with anisotropic stresses, the vanishing of $\delta G^i{}_j$ $(i \neq j)$ leads to

$$\varphi = \psi + \Gamma \quad . \tag{12}$$

Combining the $(i,0)$, $(5,0)$ and $(5,i)$ components of Einstein equations, we have the equation for W,

$$a^2 \Box_5 W = 3\Big(\frac{a''}{a} - \frac{a'^2}{a^2}\Big)W \quad , \tag{13}$$

where a dot represents the derivative with respect to u, and a prime a derivative with respect to v. The five-dimensional d'Alembert operator is now

$$\Box_5 = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^A}\Big(\sqrt{|g|}g^{AB}\frac{\partial}{\partial x^B}\Big) = \frac{1}{a^2}\Big(\partial_u^2 - \partial_v^2 - 3\frac{a'}{a}\partial_v - \nabla^2\Big) \quad .$$

The other equations, namely $(0,0)$, (i,i) and $(5,5)$ can be written as

$$
\begin{aligned}
&-3a^2\Box_5\psi + 3\ddot{\psi} - \nabla^2\psi + \nabla^2\Gamma - 3\frac{a'}{a}\Gamma' - 6\frac{a''}{a}\Gamma = 0 \quad , \\
&-a^2\Box_5\psi + a^2\Box_5\Gamma + 3\ddot{\psi} - \nabla^2\psi + \nabla^2\Gamma - 3\frac{a'}{a}\Gamma' - 6\frac{a''}{a}\Gamma = 0 \quad , \\
&-3\ddot{\psi} + \nabla^2\psi - \nabla^2\Gamma + 3\frac{a'}{a}\Gamma' + 6\frac{a'^2}{a^2}\Gamma + 6\frac{a'}{a}\phi'\delta\phi = 0 \quad .
\end{aligned}
$$

They lead to the decoupled equations

$$
a^2\Box_5(2\psi + \Gamma) = 0 \tag{14}
$$

and

$$
a^2\Box_5\Gamma = 4\Big(\frac{a''}{a} - \frac{a'^2}{a^2}\Big)\Gamma \tag{15}
$$

Equations (13), (14) and (15) can be expressed in a compact form as

$$
\Box_5\omega_i = -M_i^2\omega_i \quad , \tag{16}
$$

where we introduced the fields

$$
\omega_1 = W,\ \omega_2 = 2\psi + \Gamma,\ \omega_3 = \Gamma
$$

where $-M_i^2\ell^2 = 3, 0, 4$ for $i = 1, 2, 3$, respectively.

Together, the "constraint equations", that is $(i,0)$, $(5,0)$ and $(5,i)$ components of Einstein equations, (16) describe the complete evolution of the scalar fluctuation in coordinate system $\widetilde{X}^A$.

B. Boundary Conditions

Perturbing a (bulk) metric in the presence of a brane, we have to care about obeying the boundary conditions. In fact, any solution of Einstein Equations in the bulk, in the presence of a brane, should obey the so-called Darmois-Israel conditions [18], which in the case of a scalar interaction was discussed in [17]. Perturbations of such solutions should be subject to the above boundary conditions. For a static brane the solution is simple, because the position of the brane is fixed and generally independent of the brane position. However, when the brane is dynamical, that is, view from the point of view of the bulk, we may have

distortions of the brane quite difficult to deal with, since the position of the brane itself depends on the brane coordinates (see [19][20]).

In the present case, if we write boundary conditions in terms of $\widetilde{X}^A$, that make the bulk equations diagonal, the boundary conditions will not be diagonal. This is contrary to the usual static brane where the bulk equations and boundary conditions can be diagonalized for the same variables [12]. The complexity of the boundary conditions reflects the fact that our brane is moving in the coordinate $\widetilde{X}^A$. The boundary conditions can be diagonalized in the original coordinate $X^A = (t, x^i, y)$. It was shown that to keep the whole metric to be effectively Z_2 symmetric, the diagonal variables (or Φ, Ψ and N) should be even while the off-diagonal variable (or B) should be odd across the brane [2]. Therefore, from the perturbed Einstein equations in the coordinate system X^A, we get the boundary conditions

$$\Psi'(y_0) = -\Phi'(y_0) = \beta N(y_0) \tag{17}$$

$$B\Big|_{y=y_0} = 0 \tag{18}$$

where $\beta = \frac{2\cosh y_0}{\sinh y_0} - \frac{1}{6}\ell\sigma_0$ for a very slowly variation of the bulk scalar field. Notice that the first derivative of the even functions will give rise to a jump across the brane, thus its value may have a sign ambiguity. Here we evaluate them in the positive interval $0 \le y \le y_0$.

Combining (5) and

$$H^2 = \frac{1}{6}(\Lambda_{5,eff} + \frac{1}{6}\sigma_0^2) = -\frac{1}{\ell^2} + \frac{1}{36}\sigma_0^2 \quad ,$$

we get

$$\beta = \frac{\cosh y_0}{\sinh y_0} \quad .$$

IV. SOLUTION OF THE PERTURBED EQUATIONS

In order to find solutions which satisfy the boundary conditions at the brane, it is convenient to do the calculations of the bulk equations in coordinate X^A. This can be simply done by changing the d'Alembert operator into

$$\Box_5 = \frac{1}{\ell^2 \sinh^2 y}\Big[\frac{1}{H^2}\partial_t^2 + \frac{3}{H}\partial_t - \sinh^2 y\partial_y^2 - 4\sinh y\cosh y\partial_y - e^{-2Ht}\nabla^2\Big] \quad .$$

Separating (16) by

$$\omega = \int \mathrm{d}\lambda\, u_\lambda(y)\omega_\lambda(t, \mathbf{x}) \quad ,$$

we have the equations

$$u''_\lambda + 4\coth y\, u'_\lambda + (-M_i^2\ell^2 + \frac{\lambda^2}{\sinh^2 y})u_\lambda = 0 \tag{19}$$

and

$$H^2(\Box_4 + \lambda^2)\omega_\lambda = \ddot{\omega}_\lambda + 3H\dot{\omega}_\lambda + (-e^{-2Ht}\nabla^2 + \lambda^2)H^2\omega_\lambda = 0 \quad . \tag{20}$$

Above, λ^2 is the separation constant corresponds to the Kaluza-Klein (KK) mass which appears in the KK compactification. The general solution for equation (20) is

$$\omega_\lambda = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3p \;\; (-\eta)^{3/2}\mathcal{H}_\nu(-p\eta)e^{ip\mathrm{x}} \quad ,$$

where $\nu^2 = (9/4) - \lambda^2$, and we have expressed it with the conformal time $\eta = -e^{-Ht}$. $\mathcal{H}_\nu$ is the arbitrary combination of the Hankel functions $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$.

The general solution for equation (19) is

$$u_\lambda(y) = A_1\frac{P^\nu_{\mu-1/2}(\cosh y)}{\sinh^{3/2} y} + A_2\frac{Q^\nu_{\mu-1/2}(\cosh y)}{\sinh^{3/2} y} \quad ,$$

where $\mu_i = 1, 2, 0$ for $i = 1, 2, 3$, respectively.

In the coordinate system $\widetilde{X}^A$ the solutions ω_i should satisfy “constraint equations”, that is the $(i, 0)$, $(5, i)$ and $(5, 0)$ components of Einstein equations. Expressing the first two constraint equations, namely the $(i, 0)$ and $(5, i)$ components of Einstein equations in terms of coordinates $X^A(t, x^i, y)$, we have

$$\cosh y \sinh y\omega'_1 - 2\sinh y\omega'_2 + 3\sinh^2 y\omega_1 = \frac{1}{H}(\dot{\omega}_1 - 2\cosh y\dot{\omega}_2) \quad , \tag{21}$$

$$\sinh y\omega'_1 + \cosh y\sinh y(-3\omega'_3 + \omega'_2) - 6\sinh^2 y\omega_3 = \frac{1}{H}(\cosh y\dot{\omega}_1 - 3\dot{\omega}_3 + \dot{\omega}_2) \quad . \tag{22}$$

where a dot (or a prime) donates the derivative with t (or y).

If we get a set of solutions of ω_i, the results for (Φ, Ψ, B, N) can be obtained immediately through relations (10). In the rest of this section, we focus on the possible solutions for ω_i.

Constraint (21) involves only ω_1 and ω_2. With the property of the Hankel function, one notices that the derivative with respect to the conformal time η will give rise to a term with a factor $p\eta$ which cannot be cancelled by terms on the left side of (21). Therefore, the solutions for ω_1 and ω_2 should contain the $\mathcal{H}_{\nu-2}$ term as well as the $\mathcal{H}_\nu$ term. We assume that

$$\omega_1(\nu) = C_1\frac{R^\nu_{1/2}(\cosh y)}{\sinh^{3/2} y}(-\eta)^{3/2}\mathcal{H}_\nu(-p\eta) + D_1\frac{R^{\nu-2}_{1/2}(\cosh y)}{\sinh^{3/2} y}(-\eta)^{3/2}\mathcal{H}_{\nu-2}(-p\eta) \quad ,$$

$$\omega_2(\nu) = C_2\frac{R^\nu_{3/2}(\cosh y)}{\sinh^{3/2} y}(-\eta)^{3/2}\mathcal{H}_\nu(-p\eta) + D_2\frac{R^{\nu-2}_{3/2}(\cosh y)}{\sinh^{3/2} y}(-\eta)^{3/2}\mathcal{H}_{\nu-2}(-p\eta) \quad ,$$

where R_μ^ν is an arbitrary combination of associative Legendre function P_μ^ν and Q_μ^ν.

With the constraint (21), we found that the coefficients should satisfy (see Appendix(B 1) for a detailed derivation)

$$\begin{aligned}
C_1(\nu) &= h_\nu \quad , \\
D_1(\nu) &= (\frac{5}{2}-\nu)(\nu-\frac{1}{2})h_\nu \quad , \\
C_2(\nu) &= \frac{1}{2}h_\nu \quad , \\
D_2(\nu) &= \frac{1}{2}(\frac{5}{2}-\nu)(\frac{7}{2}-\nu)h_\nu \quad ,
\end{aligned} \tag{23}$$

where h_ν is an arbitrary value.

However, not all the solutions that satisfy the constraint (21) can meet the requirement (22). One is always allowed to expand the target solutions in the series of solutions that satisfy (21), that is

$$\begin{aligned}
\omega_1 &= \sum_{n=-\infty}^{n=+\infty} a_{\nu_0+2n}\frac{R_{1/2}^{\nu_0+2n}}{\sinh^{3/2}y}(-\eta)^{3/2}\mathcal{H}_{\nu_0+2n}(-p\eta) = \sum_{n=-\infty}^{n=+\infty}\omega_1(\nu_0+2n) \quad , \\
\omega_2 &= \sum_{n=-\infty}^{n=+\infty} b_{\nu_0+2n}\frac{R_{3/2}^{\nu_0+2n}}{\sinh^{3/2}y}(-\eta)^{3/2}\mathcal{H}_{\nu_0+2n}(-p\eta) = \sum_{n=-\infty}^{n=+\infty}\omega_2(\nu_0+2n) \quad , \\
\omega_3 &= \sum_{n=-\infty}^{n=+\infty} c_{\nu_0+2n}\frac{R_{-1/2}^{\nu_0+2n}}{\sinh^{3/2}y}(-\eta)^{3/2}\mathcal{H}_{\nu_0+2n}(-p\eta)
\end{aligned} \tag{24}$$

where ν_0 is arbitrary and $\omega_1(\nu)$ and $\omega_2(\nu)$ are defined as above with coefficients determined as in (23). Note that here R_μ^ν is not the same as that in $\omega_i(\nu)$ since the coefficients of P_μ^ν and Q_μ^ν may be different.

By imposing the constraint (22), we found that (see Appendix (B 2))

$$\begin{aligned}
a_\nu &= -\frac{4\nu h_\nu}{(\frac{5}{2}+\nu)(\frac{3}{2}-\nu)} \\
b_\nu &= \frac{2\nu h_\nu}{(\frac{5}{2}+\nu)} \quad , \\
c_\nu &= -\frac{2}{3}h_\nu\frac{\nu(\frac{1}{2}+\nu)}{(\frac{5}{2}+\nu)(\frac{3}{2}-\nu)} \quad ,
\end{aligned} \tag{25}$$

where h_ν is defined by the recursion relation

$$\frac{h_\nu}{h_{\nu+2}} = -(\frac{1}{2}-\nu)(\frac{3}{2}-\nu)\frac{\frac{5}{2}+\nu}{\frac{5}{2}-\nu} \quad . \tag{26}$$

Although we have not yet imposed any constraint on the effective mass λ^2 (related to the index ν), one can easily find that in order to make the solutions physically appropriate we should remove the divergent part of the summation (24). A cutoff will occur when the recursion meets zero. There are only two possible solutions.

In one case, we find from the relation (26) that $h_{1/2} = 0$ when $\nu_0 = 1/2$. This leads to the fact that all h_ν with $\nu = \nu_0 - 2|n|$ will vanish. By inverting the relation (26), one finds that the terms above $h_{5/2}$ with $\nu = 5/2 + 2|n|$ also vanish. The only nonzero term in the series expansion is $\nu = 5/2$.

In another case, the series is safely cutoff at $\nu_0 = 3/2$. $h_{3/2} = 0$ (so are the lower terms) if $h_{7/2}$ is finite. The terms above $h_{7/2}$ are all physical. It should be noted that $a_{3/2}$ and $c_{3/2}$ will not diverge for $h_{3/2} = 0$. The solutions contain infinite terms and have the lowest index at $\nu_0 = 3/2$.

We can express the solutions explicitly. In the first case

$$\begin{aligned} \omega_1 &= 2h \, \frac{R_{1/2}^{5/2}}{\sinh^{3/2} y} (-\eta)^{3/2} \mathcal{H}_{5/2}(-p\eta) \quad , \\ \omega_2 &= h \frac{R_{3/2}^{5/2}}{\sinh^{3/2} y} (-\eta)^{3/2} \mathcal{H}_{5/2}(-p\eta) \quad , \\ \omega_3 &= h \frac{R_{-1/2}^{5/2}}{\sinh^{3/2} y} (-\eta)^{3/2} \mathcal{H}_{5/2}(-p\eta) \quad , \end{aligned} \tag{27}$$

In the second case

$$\begin{aligned} \omega_1 &= \sum_{k=0}^{+\infty} -h \frac{\frac{16}{3}(\frac{3}{2} + 2k)}{(2k-1)(2k)!![2(k+2)]!!} \; \frac{R_{1/2}^{3/2+2k}}{\sinh^{3/2} y} (-\eta)^{3/2} \mathcal{H}_{3/2+2k}(-p\eta) \quad , \\ \omega_2 &= \sum_{k=0}^{+\infty} -h \frac{\frac{8}{3}(\frac{3}{2} + 2k)}{(2k-1)[2(k-1)]!![2(k+2)]!!} \; \frac{R_{3/2}^{3/2+2k}}{\sinh^{3/2} y} (-\eta)^{3/2} \mathcal{H}_{3/2+2k}(-p\eta) \quad , \\ \omega_3 &= \sum_{k=0}^{+\infty} -h \frac{\frac{8}{9}(\frac{3}{2} + 2k)(2k+2)}{(2k-1)(2k)!![2(k+2)]!!} \; \frac{R_{-1/2}^{3/2+2k}}{\sinh^{3/2} y} (-\eta)^{3/2} \mathcal{H}_{3/2+2k}(-p\eta) \quad , \end{aligned} \tag{28}$$

where h is arbitrary and we have defined

$$(2n)!! = \prod_{k=1}^{n} (2k) \quad ,$$

with the convention that $0!! = 1$ and $(-2)!! = \infty$.

Notice that ω_1 is the solution of the combination of constraint equations, so the other constraint given by $(5, 0)$ equation will be satisfied automatically. These are all possible

solutions for ω_i satisfying all the constraint equations. Hence they are the exact solutions for the Einstein equations.

The remaining task is to test whether the solutions obtained can satisfy the boundary condition to make the metric effectively Z_2 symmetric. We will restrict ourselves in the case $H\ell \ll 1$, or equivalently $\sinh y_0 \sim \cosh y_0 \gg 1$.

In (24)

$$R^{\nu}_{\mu}(y) = P^{\nu}_{\mu}(y) - \alpha_\nu Q^{\nu}_{\mu}(y)$$

where $\mu = 1/2, 3/2, -1/2$. The coefficient α_ν will be determined from the condition (18).

It is worthwhile mentioning that in the background we are interested in, the solutions indeed satisfy condition (17), thus the perturbed background is effectively Z_2-symmetric (see Appendix (C)).

V. SUMMARY AND DISCUSSIONS

In this paper, we derived the exact analytic solutions for scalar perturbations in a brane world model with curved de Sitter brane. Due to the existence of the bulk scalar field, the scalar perturbations are quite complicated. We found a coordinate system which can make the perturbation equations in the bulk diagonal. We have obtained the analytic solutions satisfying boundary conditions and other constraint equations.

In order to make the result physically legal, we have to cut off the divergent part of the solutions by restricting the index ν. This leads to two results: for the first, $\nu = 5/2$, the series contain only one term as shown in (27); for the second case, the solution is a combination of infinite waves as shown in (28) and the index has a lowest value at $\nu = 3/2$. In both cases the index of the waves, ν, is real. Considering $\nu^2 = 9/4 - \lambda^2$, the requirement $\nu \geq 3/2$ leads to the KK mass $\lambda^2 \leq 0$, implying the the solution involving the tachyonic mode is instable. This result agrees to the numerical argument, that different from the flat brane case, for the de Sitter branes the square of the radion mass is not positive, which leads to a strong tachyonic instability [14][15]. Our exact analytic result can be used as a support to these argument.

The tachyonic instability for inflating branes means that the braneworlds with inflation driven by the bulk inflaton are hard to stabilize. It is of interest to study what kind of mechanism can be introduced to stabilize the brane world. One expectation is that the

stabilization can be achieved by introducing another scalar field on the brane in addition to the one in the bulk [15]. Whether this attempt can work needs further investigation.

APPENDIX A: SCALAR PERTURBATION OF THE METRIC

In the pseudo spacetime $\widetilde{X}^A = (u, x^i, v)$,

$$
\begin{aligned}
a^2\delta G^0{}_0 &= \Big[2\nabla^2 + 3\frac{\partial^2}{\partial v^2} + 9\frac{a'}{a}\frac{\partial}{\partial v}\Big]\psi + \Big[\nabla^2 - 3\frac{a'}{a}\frac{\partial}{\partial v} - 6\frac{a''}{a}\Big]\Gamma \\
a^2\delta G^i{}_j &= \Big[\nabla^2 - \frac{\partial^2}{\partial v^2} - 3\frac{a'}{a}\frac{\partial}{\partial v}\Big]\varphi + \Big[-\nabla^2 + 2\frac{\partial^2}{\partial v^2} + 6\frac{a'}{a}\frac{\partial}{\partial v} - 2\frac{\partial^2}{\partial u^2}\Big]\psi \\
&\quad + \Big[-\nabla^2 - 3\frac{a'}{a}\frac{\partial}{\partial v} - 6\frac{a''}{a} - \frac{\partial^2}{\partial u^2}\Big]\Gamma + \Big[-\frac{\partial^2}{\partial u\partial v} - 3\frac{a'}{a}\frac{\partial}{\partial u}\Big]W \\
&\qquad (i = j) \\
a^2\delta G^i{}_j &= \partial_i\partial_j(\varphi - \psi - \Gamma) \qquad\qquad (i \neq j) \\
a^2\delta G^5{}_5 &= \Big[-\nabla^2 - 3\frac{a'}{a}\frac{\partial}{\partial v}\Big]\varphi + \Big[2\nabla^2 + 9\frac{a'}{a}\frac{\partial}{\partial v} - 3\frac{\partial^2}{\partial u^2}\Big]\psi \\
&\quad -3\frac{a'}{a}\dot{W} - 12\frac{a'^2}{a^2}\Gamma \\
a^2\delta G^i{}_0 &= \partial_i\Big(-\frac{W'}{2} - \frac{3}{2}\frac{a'}{a}W - 2\dot{\psi} - \dot{\Gamma}\Big) \\
a^2\delta G^5{}_0 &= \frac{1}{2}\nabla^2 W - 3\dot{\psi}' + 3\frac{a'}{a}\dot{\Gamma} \\
a^2\delta G^5{}_i &= \partial_i\Big(\frac{\dot{W}}{2} + \varphi' - 2\psi' + 3\frac{a'}{a}\Gamma\Big)
\end{aligned}
$$

where a dot represents one derivative over u, while a prime over v.

APPENDIX B: MATCHING THE CONSTRAINTS

1. Matching (21)

The following recurrence relations of Legendre functions [21] are useful for the derivations

$$R_\mu^{\nu+1}(z) = (z^2-1)^{-1/2}\Big[(\mu-\nu)z\,R_\mu^\nu(z) - (\mu+\nu)R_{\mu-1}^\nu(z)\Big] \tag{B1}$$

$$(\mu-\nu+1)R_{\mu+1}^\nu(z) = (2\mu+1)zR_\mu^\nu(z) - (\mu+\nu)R_{\mu-1}^\nu(z) \tag{B2}$$

$$(\mu-\nu+1)(z^2-1)^{1/2}R_\mu^{\nu-1} = zR_\mu^\nu - R_{\mu-1}^\nu \tag{B3}$$

Note that the recurrences are both valid for P_μ^ν and Q_μ^ν, so we simply use the notation R_μ^ν to denote the former or the latter or the linear combination of them.

With the properties of the Hankel functions we have

$$\frac{\mathrm{d}}{\mathrm{d}t}((-\eta)^{3/2}\mathcal{H}_\nu(-p\eta)) = H\eta\Big[p(-\eta)^{3/2}\mathcal{H}_{\nu-1} + (\frac{3}{2}-\nu)(-\eta)^{1/2}\mathcal{H}_\nu\Big] \quad ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}((-\eta)^{3/2}\mathcal{H}_{\nu-2}(-p\eta)) = H\eta\Big[-p(-\eta)^{3/2}\mathcal{H}_{\nu-1} + (\nu-\frac{1}{2})(-\eta)^{1/2}\mathcal{H}_{\nu-2}\Big] \quad .$$

Considering the right hand side of the constraint (21), the $\mathcal{H}_{\nu-1}$ terms in $\dot\omega_1$ and $\dot\omega_2$ should be eliminated, that is (here and after we will simplify $\underline{\sinh y}(\underline{\cosh y})$ as S(C) for short)

$$\Big[C_1 R^\nu_{1/2}(C) - 2CC_2R^\nu_{3/2}(C)\Big] - \Big[D_1R^{\nu-2}_{1/2}(C) - 2CD_2R^{\nu-2}_{3/2}(C)\Big] = 0 \quad .$$

Using (B3) for the first term and (B1) for the second term, one can find the desired relations for the coefficients (23).

We can show that these relations indeed satisfy constraint equation (21). After this setting, what is left at the right hand side is

$$\dot\omega_1 - 2C\dot\omega_2 = h(\frac{5}{2}-\nu)\frac{SR^{\nu-1}_{3/2}}{S^{3/2}}(-\eta)^{3/2}\Big[(\frac{3}{2}-\nu)\mathcal{H}_\nu + (\nu-\frac{1}{2})\mathcal{H}_{\nu-2}\Big]$$

With the help of the derivative recurrence of Legendre functions [21], we can do the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}y}\Big(\frac{R^\nu_{3/2}(C)}{S^{3/2}}\Big) = -(\nu+\frac{3}{2})\frac{R^\nu_{1/2}(C)}{S^{5/2}} \quad , \tag{B4}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\Big(\frac{R^\nu_{1/2}(C)}{S^{3/2}}\Big) = -\frac{3CR^\nu_{1/2}(C)}{S^{5/2}} + (\frac{3}{2}-\nu)\frac{R^\nu_{3/2}(C)}{S^{5/2}} \quad . \tag{B5}$$

Therefore, the left hand side of (21) becomes

$$\begin{aligned}
&CS\omega_1' - 2S\omega_2' + 3S^2\omega_1 \\
&= h\frac{(-\eta)^{3/2}}{S^{3/2}}\Big\{(\frac{3}{2}-\nu)\mathcal{H}^\nu\Big[CR^\nu_{3/2} - R^\nu_{1/2}\Big] \\
&\quad +(\nu-\frac{1}{2})\mathcal{H}_{\nu-2}\Big[(\frac{7}{2}-\nu)(\frac{5}{2}-\nu)CR^{\nu-2}_{3/2} - (\frac{5}{2}-\nu)(\nu-\frac{1}{2})R^{\nu-2}_{1/2}\Big]\Big\} \quad .
\end{aligned}$$

Using (B3) and (B1), we know that this is exactly the same as that in the right hand side.

2. Matching (22)

Up to now, we have gotten the solutions satisfying (21). Thus, we can write out

$$\omega_1(\nu) = 2h_\nu\frac{R^\nu_{1/2}}{S^{3/2}}(-\eta)^{3/2}\mathcal{H}_\nu(-p\eta) + 2h_\nu(\frac{5}{2}-\nu)(\nu-\frac{1}{2})\frac{R^{\nu-2}_{1/2}}{S^{3/2}}(-\eta)^{3/2}\mathcal{H}_{\nu-2}(-p\eta) \quad ,$$

$$\omega_2(\nu) = h_\nu\frac{R^\nu_{3/2}}{S^{3/2}}(-\eta)^{3/2}\mathcal{H}_\nu(-p\eta) + h_\nu(\frac{5}{2}-\nu)(\frac{7}{2}-\nu)\frac{R^{\nu-2}_{3/2}}{S^{3/2}}(-\eta)^{3/2}\mathcal{H}_{\nu-2}(-p\eta) \quad ,$$

and

$$\omega_1(\nu+2) = 2h_{\nu+2}\frac{R^{\nu+2}_{1/2}}{S^{3/2}}(-\eta)^{3/2}\mathcal{H}_{\nu+2}(-p\eta) + 2h_{\nu+2}(\frac{1}{2}-\nu)(\nu+\frac{3}{2})\frac{R^{\nu}_{1/2}}{S^{3/2}}(-\eta)^{3/2}\mathcal{H}_{\nu}(-p\eta) \quad ,$$
$$\omega_2(\nu+2) = h_{\nu+2}\frac{R^{\nu+2}_{3/2}}{S^{3/2}}(-\eta)^{3/2}\mathcal{H}_{\nu+2}(-p\eta) + h_{\nu+2}(\frac{1}{2}-\nu)(\frac{3}{2}-\nu)\frac{R^{\nu}_{3/2}}{S^{3/2}}(-\eta)^{3/2}\mathcal{H}_{\nu}(-p\eta) \quad ,$$

where the ratio of h_ν and $h_{\nu+2}$ are left to be determined. According to (24), it is obvious that

$$a_\nu = 2h_\nu + 2h_{\nu+2}(\frac{1}{2}-\nu)(\nu+\frac{3}{2}) \quad ,$$
$$b_\nu = h_\nu + h_{\nu+2}(\frac{1}{2}-\nu)(\frac{3}{2}-\nu) \quad .$$

The right hand side of constraint equation (22) becomes

$$-\sum_\nu S^{-3/2}\Big\{(\frac{3}{2}-\nu)\Big[a_\nu C R^\nu_{1/2} + b_\nu R^\nu_{3/2} - 3c_\nu R^\nu_{-1/2}\Big](-\eta)^{3/2}\mathcal{H}_\nu + \Big[a_\nu C R^\nu_{1/2} + b_\nu R^\nu_{3/2} - 3c_\nu R^\nu_{-1/2}\Big]p(-\eta)^{5/2}\mathcal{H}_{\nu-1}\Big\} \quad .$$

The second term should be zero. With (B2), we can determine a_ν, b_ν and c_ν as in (25) and the recurrence (26) of h_ν.

Following the same procedure as discussed in Appendix (B 1), the constraint (22) is indeed satisfied.

APPENDIX C: FIXING THE BOUNDARIES

For convenience, we write (10) in term of ω_i

$$\Phi = \frac{1}{2S^2}[C^2\omega_2 + (2+C^2)\omega_3 - C\omega_1] \quad ,$$
$$N = \frac{1}{2S^2}[\omega_2 + (1+2C^2)\omega_3 - C\omega_1] \quad ,$$
$$B = \frac{H\ell}{S}[(1+C^2)\omega_1 - C\omega_2 - 3C\omega_3] \quad .$$
$$\Psi = \frac{1}{2}[\omega_2 - \omega_3]$$

From Appendix (B 2), we know the following identity will always be true

$$C\omega_1 + \omega_2 = 3\omega_3 \quad .$$

With (18) we can further find that on the brane $\omega_1(y_0) = 2C\omega_2(y_0)$. Thus, condition (17) is equal to

$$\begin{aligned}
&\frac{1}{S^2}\Big[S\omega_1 + 2C^2\omega_2'\Big]_{y=y_0} = 0 \\
&\frac{1}{S^3}\Big[\frac{1}{2}(C^2+1)\omega_1 - \frac{1}{3}S^2\omega_2 - \frac{2}{3}S^3\omega_2'\Big]_{y=y_0} = 0 \quad .
\end{aligned}$$

The background can only be stabilized if the metric perturbations keep in small scale far less than the metric, say $|\omega_i| \ll 1 \ll \sinh y_0$. With (B4), we have

$$\omega_2'(y_0) = -\frac{1}{S}\Big[\sum_\nu (\nu+\frac{3}{2})b_\nu \frac{R^\nu_{1/2}}{S^{3/2}}(-\eta)^{3/2}H_\nu(-p\eta)\Big]_{y=y_0} \sim 0 \quad ,$$

(where for (27) ν is only 5/2, while in (28) ν runs from 3/2 to infinite). Thus the boundary conditions are satisfied. In other words, the perturbed background is also effectively Z_2-symmetric.

ACKNOWLEDGEMENT: This work was partially supported by NNSF of China, Ministry of Education of China and Ministry of Science and Technology of China under Grant NKBRSFG19990754. The work of E. Abdalla was supported by FAPESP and CNPQ, Brazil.

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[1] F. Quevedo, *Class. Quant. Grav.* **19** (2002) 5721; P. Brax and C. Bruck, *Class. Quant. Grav.* **20** (2003) R201.

[2] C. Bruck, M. Dora, R. H. Brandenberger, *Phys. Rev.* **D 62**, 123515 (2000).

[3] V. Bozza, M. Gasperini and G. Veneziano, *Nucl. Phys.* **B619**, 191(2001).

[4] T. Tanaka and X. Montes, *Nucl. Phys.* **B 582** (2000) 259.

[5] C. Csaki, M. Graesser, L. Randall and J. Terning, *Phys. Rev.* **D 62**, 045015 (2000).

[6] C. Csaki, J. Erlich, T. J. Hollowood and T. Shirman, *Nucl. Phys.* **B 581** (2000) 309.

[7] D. Langlois, R. Maartens and D. Wands, *Phys. Lett.* **B 489** (2000) 259; A. Frolov and L. Kofman, hep-th/0209133; R. Easter, D. Langlois, R. Maartens and D. Wands, hep-th/0308078; T. Hiramatsu, K. Koyama and A. Taraya, hep-th/0308072.

[8] S. Mukohyama, *Phys. Rev.* **D 62**, 084015 (2000); H. Kodama, A. Ishibashi and O. Sato, *Phys. Rev.* **D 62**, 064022 (2000); D. Langlois, *Phys. Rev.* **D 62**, 126012 (2000); K.Koyama and J. Soda, *Phys. Rev.* **D 62**, 123502 (2000).

[9] S. Kobayashi, K. Koyama and J. Soda, *Phys. Lett.* **B 501** (2001) 157; Y. Himemoto and M. Sasaki, *Phys. Rev.* **D 64** (2001) 083511.

[10] Y. Himemoto, T. Tanaka and M. Sasaki, *Phys. Rev.* **D 65**, (2002) 104020; Y. Himemoto and M. Sasaki, *Phys. Rev.* **D 63**, 044015 (2002); B. Wang, L. H. Xue, X. Zhang and W-Y P. Hwang, *Phys. Rev.* **D 67**, 123519 (2003); D. Langlois and M. Sasaki, *Phys. Rev.* **D 68**, 064012 (2003).

[11] M. Minamitsuji, Y. Himemoto and M. Sasaki, *Phys. Rev.* **68** (2003) 024016.

[12] K. Koyama and K. Takahashi, hep-th/0307073.

[13] J. Garriga and M. Sasaki, *Phys. Rev.* **D 62**, 043523 (2000).

[14] J. Martin, G. N. Felder, A. Frolov, M. Peloso and L. Kofman, hep-th/0309001.

[15] A. Frolov and L. Kofman, hep-th/0309002.

[16] H. A. Chamblin and H. S. Reall, *Nucl.Phys.* **B562** (1999) 133; [hep-th/9903225].

[17] Elcio Abdalla and Bertha Cuadros-Melgar, *Phys. Rev.* **D67** (2003) 084012, hep-th/0209101.

[18] C. Lanczos, *Phys. Zeits.* **23** (1922) 539; *Ann. der Phys.* **74** (1924) 518.W. Israel, *Nuovo Cimento* **44B** (1966) 1, Erratum: **48B** (1967) 2.

[19] G. Darmois, *Mem. Sciences Math.* **XXV** (1927) ch. V.

[20] E. Abdalla and A. C. Casali hep-th/0208008

[21] M. Abramovicz and I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1972.